Stress Analysis

- Stress Analysis is accomplished by measuring deformation of part under load and inferring the existing state of stress from measured deflection.
- Normal Stress:
  \[ \sigma_a = \frac{F_n}{A_c} \]
  where \( F_n \) = Tension Force, \( A_c \) = Area, \( \sigma_a \) = axial strain

- Ratio of change in length of the rod to the original length is axial strain:
  \[ \varepsilon_a = \frac{\delta L}{L} \]
  where \( \varepsilon_a \) = average strain, \( \delta L \) = change in length, \( L \) = original unloaded length
  - Lateral strain: \( \varepsilon_L = \frac{\delta d}{d} \)

- Strain is a small quantity, usually reported in units of \( 10^{-6} \) in/in or \( 10^{-6} \) m/m.
Hooke's Law

- Stress-strain diagrams are very important in understanding the behavior of a material under load.
- The range of stress over which this linear relationship holds is called the elastic region.
- The relationship between uniaxial stress and strain for this elastic behavior is expressed as:
  \[
  \sigma = E\epsilon
  \]
  - \(E\) is the modulus of elasticity or Young's modulus
  - The relationship is called Hooke's Law
  - \(30 \times 10^6\) PSI for steel

Consider the elongation of the rod shown in Figure 11.1, which occurs as a result of the load \(F_N\).
- As the rod is stretched in the axial direction, the cross-sectional area must decrease, since the total mass (or volume for constant density) must be conserved.
In the elastic range, there is a constant rate of change in the lateral strain as the axial strain increases. The ratio of lateral strain to axial strain is known as Poisson’s ratio:

\[ v_p = \frac{\text{lateral strain}}{\text{axial strain}} \]

\[ v_p = \varepsilon_L / \varepsilon_A \]

Engineering components are seldom subject to one-dimensional axial loading. The relationship between stress and strain must be generalized to a multi-dimensional case. Consider a 2-D geometry subject to tensile loads in both the x and y directions, resulting in normal stresses, \( \sigma_x \) and \( \sigma_y \).
Poisson's Ratio

- Consider 2 dimensional geometry, subject to tensile loads in both x and y.

\[ \varepsilon_y = \frac{\sigma_y}{E_m} - V_p \frac{\sigma_x}{E_m} \quad \varepsilon_x = \frac{\sigma_x}{E_m} - V_p \frac{\sigma_y}{E_m} \]

\[ \sigma_x = \frac{E_m(\varepsilon_x + V_p\varepsilon_y)}{1 - V_p^2} \quad \sigma_y = \frac{E_m(\varepsilon_y + V_p\varepsilon_x)}{1 - V_p^2} \]

- All stress and strain components lie in the same plane.
- Since strain is measured on the surface, this gives us information on the stress on the surface.

Resistance Strain Gauges

- Strain can be measured by methods as simple as observing the change in the distance between two scribe marks on the surface of a load-carrying member, or as advanced as optical holography.

Ideal Instrument to Measure Strain

- Good spatial resolution
- Unaffected by ambient condition change
- High frequency response for dynamic strain

- A sensor that closely meets these characteristics is the bonded resistance strain gauge.
Resistance Strain Gauges

- The resistance of a strain gauge changes when it is deformed, and this is easily related to the local strain.
- Both metallic and semiconductor materials experience this change in electrical resistance when they are subjected to a strain.

Metallic Gauges

- Consider a conductor having a uniform cross-sectional area, $A_c$, and a length, $L$, made of a material having a resistivity, $\rho_e$.
- For electrical conductor:
  \[ R = \frac{\rho_e L}{A_c} \]
  where $\rho_e$ = resistivity
- If the conductor is subjected to normal axial stress, the cross-sectional area and length changes, which changes the resistance.
  Consider total differential:
  \[ dR = \frac{(A_c(\rho_e dL + Ld\rho_e) - \rho_e L dA_c)}{A_c^2} \]

Poisson’s Ratio

- \[ \frac{dR}{R} = \frac{dL}{L}(1 + 2\nu_p) + \frac{d\rho_e}{\rho_e} \]
- Changes in resistance caused by:
  1. Change in geometry
  2. Change in value of resistivity
- Piezoresistance coefficient $\pi_1$
  \[ \pi_1 = \frac{1}{E_m} \left( \frac{d\rho_e}{\rho_e} \right) \] (Note: The change in resistivity on mechanical strain is called Piezoresistance)
- Then change in resistance can be expressed:
  \[ \frac{dR}{R}/(dL/L) = 1 + 2\nu_p + \pi_1 E_m \]
Strain Gauge

A single straight conductor is normally not practical for strain measurement with meaningful resolution. A simple solution is to bend the wire conductor so that several lengths of wire are oriented along the axis of the strain gauge.

The next slide shows the construction of a typical metallic-foil bonded strain gauge. Such a strain gauge consists of a metallic foil pattern that is formed in a manner similar to the process used to produce printed circuits. This photoetched metallic foil pattern is mounted on a plastic backing material.
The gauge length is an important specification for a particular application. Since strain is usually measured at the location on a component where the stress is a maximum and the stress gradients are high, the strain gauge averages the measured strain over the gauge length.

The variety of conditions encountered in particular applications require special construction and mounting techniques, including design variations in the backing material, the grid configuration, bonding techniques, and total gauge electrical resistance.
Strain Gauge

"If a strain gauge is placed in a state of stress in which there are both normal and lateral (transverse) components of strain, the total change in resistance of the gauge will be a result of both the axial and lateral components of strain. Strictly speaking, the relationship for change in resistance expressed in equation 11.8 is true only for a single conductor in a state of uniaxial tension or compression. The goal of the design of electrical resistance strain gauges is to create a sensor that is sensitive only to strain along one axis, and therefore has zero sensitivity to lateral and shearing strains. Actual strain gauges are sensitive to lateral strains to some degree; for most gauges, the sensitivity to shearing strains may be neglected."

Lateral Sensitivity

- The change in resistance of a strain gauge is normally expressed in terms of an empirically determined parameter called the gauge factor:

\[ \text{GF} = \frac{\delta R/R}{\delta L/L} \]  

- The gauge factor is dependent on the Poisson ratio for the gauge material and its piezoresistivity.
- For metallic strain gauges, the Poisson ratio is about 0.3 and the resulting gauge factor is ~2.
- The gauge factor represents the total change in resistance for a strain gauge, under a calibration loading condition.

**Strain Gauge Electrical Circuits**

- A Wheatstone bridge is generally used to detect the small changes in resistance that are the output of a strain gauge measurement circuit.
- Equipment is commercially available that can measure changes in gauge resistance of less than 0.0005 Ω (0.000001 µε).
- A simple strain gauge Wheatstone bridge circuit is shown next.
Strain Gauge Electrical Circuits

- Bridge output:
  \[ E_o + \delta E_o = E_o \left[ \frac{(R_1 + \delta R)R_4 - R_3R_2}{(R_1 + \delta R + R_2)(R_3 + R_4)} \right] \]

- If bridge is initially balanced so \( E_o = 0 \), and all the fixed resistors and strain gauge resistance are initially equal:
  \[ \frac{\delta E_o}{E_i} = \frac{\delta R}{R} \approx \frac{\delta R}{R} \]

- \( E_i \) = bridge output; \( \delta E_o \) = bridge deflection
- \( \delta R \) = strain gauge resistance change

Strain Gauge Electrical Circuits

- Using the relationship that \( \delta R/R = GF\varepsilon \):
  \[ \frac{\delta E_o}{E_i} = \frac{GF\varepsilon}{4 + 2(GF\varepsilon)} \approx \frac{GF\varepsilon}{4} \]
The Wheatstone bridge has several distinct advantages for use with electrical resistance gauges. The bridge may be balanced by changing the resistance of one arm of the bridge. Therefore, once the gauge is mounted in place on the test specimen under a condition of zero loading, the output from the bridge may be zeroed.

There are two schemes for circuits to accomplish this balancing.

Shunt balancing is the best, since changes in resistances are small. The strategic placement of multiple gauges in a Wheatstone bridge can both increase the bridge output and cancel out certain ambient effects and unwanted components of strain.
The output from a bridge circuit can be increased by the appropriate use of more than one active strain gauge. The use of multiple gauges can be used to compensate for unwanted effects, such as temperature or specific strain components. Consider the case when all four resistances in the bridge circuit represent active strain gauges.

In general, the bridge output is given by:

$$E_o = E\left[\frac{R_1}{R_1 + R_2} + \frac{R_3}{R_3 + R_4}\right]$$

If these gauges are now subjected to strains such that the resistances change by \(dR_i\), where \(i = 1, 2, 3, \text{ and } 4\), then the change in the bridge output voltage can be expressed by:

$$dE_o = \sum_{i=1}^{4} \frac{\partial E_o}{\partial R_i} dR_i$$

Evaluating the appropriate partial derivatives from Eq 11.17 yields:

$$dE_o = E\left[\frac{R_1 dR_1 - R_3 dR_2}{(R_1 + R_2)^2} + \frac{R_3 dR_3 - R_1 dR_4}{(R_3 + R_4)^2}\right]$$
Multiple Gauge Bridge

Assuming \( dR_i \ll R_i \), the resulting change in output voltage \( \delta E_o \) may now be expressed as \( \delta E_o \) where:

\[
\delta E_o = E_i \left( \frac{R_1 \cdot R_2}{(R_1 + R_2)^2} (\varepsilon_1 G_F - \varepsilon_2 G_F) + \frac{R_3 \cdot R_4}{(R_3 + R_4)^2} (\varepsilon_3 G_F - \varepsilon_4 G_F) \right)
\]

If \( R_1 = R_2 = R_3 = R_4 \), then

\[
\delta E_o / E_i = \frac{1}{4} (\varepsilon_1 G_F - \varepsilon_2 G_F + \varepsilon_3 G_F - \varepsilon_4 G_F)
\]

Multiple Gauge Bridge

It is possible to purchase matched sets of strain gauges for a particular application, so that \( G_F_1 = G_F_2 = G_F_3 = G_F_4 \), and:

\[
\frac{\delta E_o}{E_i} = G_F \left( \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 \right)
\]

Multiple Gauge Bridge

The last equation shows that for a bridge containing one or more active strain gauges, equal strains on opposite bridge arms sum, whereas equal strains on adjacent arms of the bridge cancel.

These characteristics can be used to increase the output of the bridge, to provide temperature compensation, or to cancel unwanted components of strain.
Bridge Constant

- Commonly used strain gauge-bridge arrangements may be characterized by a bridge constant, \( \kappa \), defined as the ratio of the actual bridge output to that output of a single gauge sensing the maximum strain, \( \varepsilon_{\text{max}} \) (assuming the remaining bridge resistances remain fixed).

Apparent Strain and Temperature Compensation

- Apparent strain is manifested as any change in gauge resistance that is not due to the component of strain being measured.
- Techniques for accomplishing temperature compensation, eliminating certain components of strain, and increasing the value of the bridge constant can be devised.
- The bridge constant is influenced by (1) the location of strain gauges on the test specimen and (2) the gauge connection positions in the bridge circuit.

Apparent Strain and Temperature Compensation

- A component of strain can be removed (compensation) from the measured signal.
- Consider a beam having a rectangular cross-section and subject to the loading condition, shown in the next slide, where the beam is subject to an axial load \( F_N \) and a bending moment, \( M \).
To remove the effects of bending strain, identical strain gauges are mounted to the top and bottom of the beam. They are connected to bridge locations 1 and 4 (opposite bridge arms). The gauges experience equal but opposite bending strains and both strain gauges are subject to the same axial strain caused by $F_N$.

The bridge output under these conditions is:

$$\frac{\delta E_o}{E_i} = \frac{GF}{4} (\varepsilon_1 + \varepsilon_4)$$

Where $\varepsilon_1 = \varepsilon_{a1} + \varepsilon_{b1}$ and $\varepsilon_4 = \varepsilon_{a4} - \varepsilon_{b4}$, with subscripts $a$ and $b$ referring to axial and bending strain, respectively.
Apparent Strain and Temperature Compensation

Hence, the bending strains cancel but the axial strain will sum, giving:

\[ \delta E_o = \frac{GF}{E_i} = \frac{2}{E_a} \]