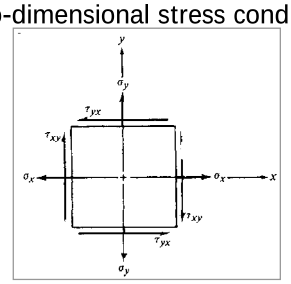
Combined Stresses and Mohr's Circle

Material in this lecture was taken from chapter 4 of Mott, Machine Elements in Mechanical Design, 2003

General Case of Combined Stresses

- Two-dimensional stress condition



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General Case of Combined Stresses con't

- The normal stresses, σ_x and σ_y, could be due to a direct tensile force or to bending. If the normal stresses were compressive (negative), the vectors would be pointing in the opposite sense, into the stress element.
- The shear stress could be due to direct shear, torsional shear, or vertical shear stress. The double-subscript notation helps to orient the direction of shear stresses. For example, τ_{xy} indicates the shear stress acting on the element face that is perpendicular to the x-axis and parallel to the y-axis.

General Case of Combined Stresses con't

- A positive shear stress is one that tends to rotate the stress element clockwise
- In the first figure, τ_{xy} is positive and τ_{yx} is negative. Their magnitudes must be equal to maintain the element in equilibrium.
- With the stress element defined, the objectives of the remaining analysis are to determine the maximum normal stress, and the planes on which these stresses occur.

Maximum Normal Stresses

- The combination of the applied normal and shear stresses that produces the maximum normal stress is called the maximum principle stress, σ_1.

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

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Maximum Normal Stresses

- The minimum principle stress, σ_2 equals:

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

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Maximum Normal Stresses con't

- The angle of inclination of the planes on which the principle stresses act, called principle planes, can be found from:

$$\phi_\sigma = \frac{1}{2}\arctan[2\tau_{xy}/(\sigma_x - \sigma_y)]$$

Measured from the positive X axis

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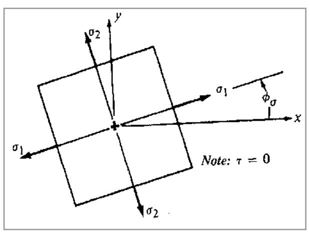
Maximum Normal Stresses con't

- The angle ϕ_σ is measured from the positive x-axis of the original stress element to the maximum principle stress, σ_1. Then the minimum principle stress, σ_2, is on the plane 90° from σ_1.



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Maximum Shear Stress

- On a different orientation of the stress element , the maximum shear stress will occur.

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

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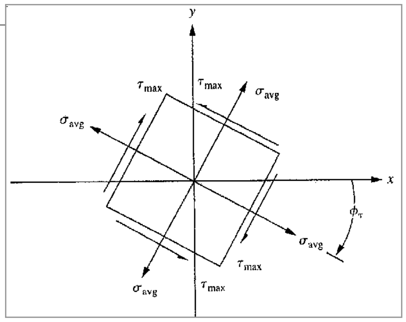
Maximum Shear Stress



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Maximum Shear Stress

- The angle of inclination of the element on which the maximum occurs is computed as follows:

$$\phi_\tau = \frac{1}{2}\arctan\left[-(\sigma_x - \sigma_y)/2\tau_{xy}\right]$$

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Maximum Shear Stress

- The angle between the principle stress element and the maximum shear stress element is always 45º.
- On the maximum shear stress element, there will be normal stresses of equal magnitude acting perpendicular to the planes on which the maximum shear stresses are acting.

Average Normal Stress

- The average of two applied normal stresses:

$$\sigma_{avg} = (\sigma_x + \sigma_y)/2$$

General Procedure for Analyzing any Combined Stress

1. Decide for which point you want to compute the stresses.
2. Clearly specify the coordinate system for the object, the free-body diagram, and the magnitude and direction of forces.
3. Compute the stresses on the selected point due to the applied forces, and show the stresses acting on a stress element at the desired point with careful attention to directions. Figure 4–3 is a model for how to show these stresses.
4. Compute the principal stresses on the point and the directions in which they act. Use Equations (4–1), (4–2), and (4–3).

General Procedure

5. Draw the stress element on which the principal stresses act, and show its orientation relative to the original x-axis. It is recommended that the principal stress element be drawn beside the original stress element to illustrate the relationship between them.
6. Compute the maximum shear stress on the element and the orientation of the plane on which it acts. Also, compute the normal stress that acts on the maximum shear stress element. Use Equations (4–4), (4–5), and (4–6).
7. Draw the stress element on which the maximum shear stress acts, and show its orientation to the original x-axis. It is recommended that the maximum shear stress element be drawn beside the maximum principal stress element to illustrate the relationship between them.
8. The resulting set of three stress elements will appear as shown in Figure 4–6.

Example 4.1

Example Problem 4–1 The shaft shown in Figure 4–7 is supported by two bearings and carries two V-belt sheaves. The tensions in the belts exert horizontal forces on the shaft, tending to bend it in the x-z plane. Sheave B exerts a clockwise torque on the shaft when viewed toward the origin of the coordinate system along the x-axis. Sheave C exerts an equal but opposite torque on the shaft. For the loading condition shown, determine the principal stresses and the maximum shear stress on element K on the front surface of the shaft (on the positive z-side) just to the right of sheave B. Follow the general procedure for analyzing combined stresses given in this section.

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Objective Compute the principal stresses and the maximum shear stresses on element K.

Given Shaft and loading pattern shown in Figure 4–7.

Analysis Use the general procedure for analyzing combined stresses.

Example 4.1

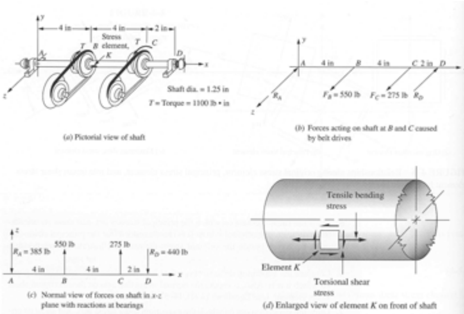
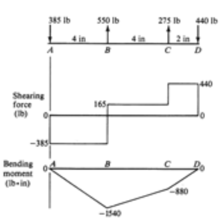


FIGURE 4–7 Shaft supported by two bearings and carrying two V-belt sheaves

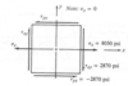
Example 4.1

FIGURE 4–8 Shearing force and bending moment diagrams for the shaft



Results Element K is subjected to bending that produces a tensile stress acting in the x-direction. Also, there is a torsional shear stress acting at K. Figure 4–8 shows the shearing force and bending moment diagrams for the shaft and indicates that the bending moment at K is 1540 lb · in. The bending stress is therefore

Example 4.1



$$\sigma_x = M/S$$
$$S = \pi D^3/32 = [\pi(1.25\ \text{in})^3]/32 = 0.192\ \text{in}^3$$
$$\sigma_x = (1540\ \text{lb}\cdot\text{in})/(0.192\ \text{in}^3) = 8030\ \text{psi}$$

The torsional shear stress acts on element K in a way that causes a downward shear stress on the right side of the element and an upward shear stress on the left side. This action results in a tendency to rotate the element in a *clockwise* direction, which is the *positive direction for shear stresses* according to the standard convention. Also, the notation for shear stresses uses double subscripts. For example, τ_{xy} indicates the shear stress acting on the face of an element that is perpendicular to the x-axis and parallel to the y-axis. Thus, for element K,

$$\tau_{xy} = T/Z_p$$
$$Z_p = \pi D^3/16 = \pi(1.25\ \text{in})^3/16 = 0.383\ \text{in}^3$$
$$\tau_{xy} = (1100\ \text{lb}\cdot\text{in})/(0.383\ \text{in}^3) = 2870\ \text{psi}$$

The values of the normal stress, σ_x, and the shear stress, τ_{xy}, are shown on the stress element K in Figure 4–9. Note that the stress in the y-direction is zero for this loading. Also, the value of the shear stress, τ_{yx}, must be equal to τ_{xy}, and it must act as shown in order for the element to be in equilibrium.

We can now compute the principal stresses on the element, using Equations (4–1) through (4–3). The maximum principal stress is

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{(4–1)}$$
$$\sigma_1 = (8030/2) + \sqrt{(8030/2)^2 + (2870)^2}$$
$$\sigma_1 = 4015 + 4935 = 8950\ \text{psi}$$

The minimum principal stress is

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{(4–2)}$$
$$\sigma_2 = (8030/2) - \sqrt{(8030/2)^2 + (2870)^2}$$
$$\sigma_2 = 4015 - 4935 = -920\ \text{psi (compression)}$$

The direction in which the maximum principal stress acts is

$$\phi_\sigma = \tfrac{1}{2}\arctan[2\tau_{xy}/(\sigma_x - \sigma_y)] \quad \text{(4–3)}$$
$$\phi_\sigma = \tfrac{1}{2}\arctan[(2)(2870)/(8030)] = 17.8°$$

The positive sign calls for a *clockwise* rotation of the element.

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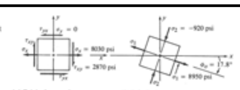
Example 4.1



(a) Original stress element (b) Principal stress element

The principal stresses can be shown on a stress element as illustrated in Figure 4–10. Note that the element is shown in relation to the original element to emphasize the direction of the principal stresses in relation to the original x-axis. The positive sign for ϕ_σ indicates that the principal stress element is rotated *clockwise* from its original position.

Now the maximum shear stress element can be defined, using Equations (4–4) through (4–6):

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{(4–4)}$$
$$\tau_{max} = \sqrt{(8030/2)^2 + (2870)^2}$$
$$\tau_{max} = \pm 4935\ \text{psi}$$

The two pairs of shear stresses, $+\tau_{max}$ and $-\tau_{max}$, are equal in magnitude but opposite in direction.

The orientation of the element on which the maximum shear stress acts is found from Equation (4–5):

$$\phi_\tau = \tfrac{1}{2}\arctan[-(\sigma_x - \sigma_y)/2\tau_{xy}] \quad \text{(4–5)}$$
$$\phi_\tau = \tfrac{1}{2}\arctan(-8030/[(2)(2870)]) = -27.2°$$

The negative sign calls for a *counterclockwise* rotation of the element.

There are equal normal stresses acting on the faces of this stress element, which have the value of

$$\sigma_{avg} = (\sigma_x + \sigma_y)/2 \quad \text{(4–6)}$$
$$\sigma_{avg} = 8030/2 = 4015\ \text{psi}$$

Figure 4–11 shows the stress element on which the maximum shear stress acts in relation to the original stress element. Note that the angle between this element and the principal stress element is 45°.

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Mohr's Circle

- Because of the many terms and signs involved, and the many calculations required in the computation of the principle stresses and the maximum shear stress, there is a rather high probability of error. Using the graphic aid Mohr's circle helps to minimize errors and gives a better "feel" for the stress condition at the point of interest.

Mohr's Circle

- After Mohr's circle has been constructed, it can be used for the following:

1. Finding the maximum and minimum principal stresses and the directions in which they act.
2. Finding the maximum shear stresses and the orientation of the planes on which they act.
3. Finding the value of the normal stresses that act on the planes where the maximum shear stresses act.
4. Finding the values of the normal and shear stresses that act on an element with any orientation.

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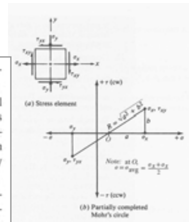
Mohr's Circle

- The data needed to construct Mohr's circle are the same as those needed to compute the preceding values, because the graphical approach is an exact analogy to the computations.

Mohr's Circle

- Mohr's circle is actually a plot of the combination of normal and shearing stresses that exist on a stress element for all possible angles of orientation of the element. This method is particularly valuable in experimental stress analysis work because the results obtained from many types of standard strain gage instrumentation techniques give the necessary inputs for the creation of Mohr's circle.

Methodology



1. Perform the stress analysis to determine the magnitudes and directions of the normal and shear stresses acting at the point of interest.
2. Draw the stress element at the point of interest as shown in Figure 4–12(a). Normal stresses on any two mutually perpendicular planes are drawn with tensile stresses positive—projecting outward from the element. Compressive stresses are negative—directed inward on the face. Note that the *resultants* of all normal stresses acting in the chosen directions are plotted. Shear stresses are considered to be positive if they tend to rotate the element in a *clockwise* (cw) direction, and negative otherwise.

   Note that on the stress element illustrated, σ_x is positive, σ_y is negative, τ_{xy} is positive, and τ_{yx} is negative. This assignment is arbitrary for the purpose of illustration. In general, any combination of positive and negative values could exist.
3. Refer to Figure 4–12(b). Set up a rectangular coordinate system in which the positive horizontal axis represents positive (tensile) normal stresses, and the positive vertical axis represents positive (clockwise) shear stresses. Thus, the plane created will be referred to as the σ-τ *plane*.
4. Plot points on the σ-τ plane corresponding to the stresses acting on the faces of the stress element. If the element is drawn in the *x*-y plane, the two points to be plotted are σ_x, τ_{xy} and σ_y, τ_{yx}.

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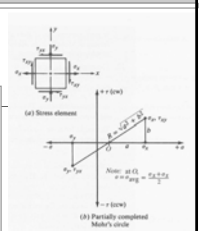
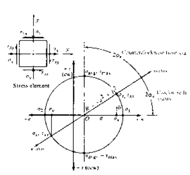
Methodology





5. Draw the line connecting the two points.
6. The resulting line crosses the σ-axis at the center of Mohr's circle at the average of the two applied normal stresses, where

$$\sigma_{avg} = (\sigma_x + \sigma_y)/2$$

   The center of Mohr's circle is called *O* in Figure 4–12.
7. Note in Figure 4–12 that a right triangle has been formed, having the sides *a*, *b*, and *R*, where

$$R = \sqrt{a^2 + b^2}$$

   By inspection, we can see that

$$a = (\sigma_x - \sigma_y)/2$$
$$b = \tau_{xy}$$

   The point labeled *O* is at a distance of $\sigma_x - a$ from the origin of the coordinate system. We can now proceed with the construction of the circle.
8. Draw the complete circle with the center at *O* and a radius of *R*, as shown in Figure 4–13.
9. The point where the circle crosses the σ-axis at the right gives the value of the maximum principal stress, σ_1. Note that $\sigma_1 = \sigma_{avg} + R$.
10. The point where the circle crosses the σ-axis at the left gives the minimum principal stress, σ_2. Note that $\sigma_2 = \sigma_{avg} - R$.

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Methodology

11. The coordinates of the top of the circle give the maximum shear stress and the average normal stress that act on the element having the maximum shear stress. Note that $\tau_{max} = R$.

Note: The following steps relate to determining the angles of inclination of the principal stress element and the maximum shear stress element in relation to the original *x*-axis. It is important to realize that angles on Mohr's circle are actually *double* the true angles. Refer to Figure 4–13; the line from *O* through the first point plotted, σ_x, τ_{xy}, represents the original *x*-axis, as noted in the figure. The line from *O* through the point σ_y, τ_{yx} represents the original *y*-axis. Of course, on the

original element, these axes are 90° apart, not 180°, illustrating the double-angle feature of Mohr's circle. Having made this observation, we can continue with the development of the process.

12. The angle $2\phi_\sigma$ is measured from the *x*-axis on the circle to the σ-axis. Note that

$$2\phi_\sigma = \arctan(b/a)$$

It is also important to note the direction *from the x-axis to the σ-axis* (clockwise or counterclockwise). This is necessary for representing the relation of the principal stress element to the original stress element properly.

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Methodology

13. The angle from the x-axis on the circle to the vertical line through τ_{max} gives $2\phi_\tau$. From the geometry of the circle, in the example shown, we can see that

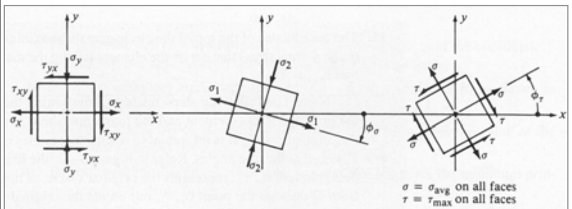
$$2\phi_\tau = 90^\circ - 2\phi_\sigma$$

Other combinations of the initial stresses will result in different relationships between $2\phi_\sigma$ and $2\phi_\tau$. The specific geometry on the circle should be used each time. See Example Problems 4–3 to 4–8 that follow this section.

Again it is important to note the direction *from the x-axis to the τ_{max}-axis* for use in orienting the maximum shear stress element. You should also note that the σ-axis and the τ_{max}-axis are always 90° apart on the circle and therefore 45° apart on the actual element.

14. The final step in the process of using Mohr's circle is to draw the resulting stress elements in their proper relation to the original element, as shown in Figure 4–14.

Display of Results from Mohr's Circle

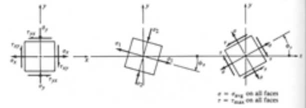


(*a*) Original stress element (*b*) Principal stress element (*c*) Maximum shear stress element

Example 4-2

Example Problem 4–2 The shaft shown in Figure 4–7 is supported by two bearings and carries two V-belt sheaves. The tensions in the belts exert horizontal forces on the shaft, tending to bend it in the x-z plane. Sheave B exerts a clockwise torque on the shaft when viewed toward the origin of the coordinate system along the x-axis. Sheave C exerts an equal but opposite torque on the shaft. For the loading condition shown, determine the principal stresses and the maximum shear stress on element K on the front surface of the shaft (on the positive z-side) just to the right of sheave B. Use the procedure for constructing Mohr's circle in this section.

FIGURE 4–14
Display of results from Mohr's circle



(a) Original stress element (b) Principal stress element (c) Maximum shear stress element

Example 4-2

Objective: Determine the principal stresses and the maximum shear stresses on element *K*.

Given: Shaft and loading pattern shown in Figure 4–7.

Analysis: Use the *Procedure for Constructing Mohr's Circle*. Some intermediate results will be taken from the solution to Example Problem 4–1 and from Figures 4–7, 4–8, and 4–9.

Results: ***Steps 1 and 2.*** The stress analysis for the given loading was completed in Example Problem 4–1. Figure 4–15 is identical to Figure 4–9 and represents the results of Step 2 of the Mohr's circle procedure.

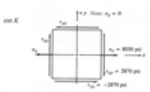
Steps 3–6. Figure 4–16 shows the results. The first point plotted was

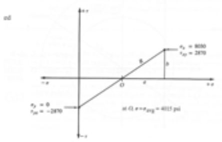
$$\sigma_x = 8030 \text{ psi}, \tau_{xy} = 2870 \text{ psi}$$

The second point was plotted at

$$\sigma_y = 0 \text{ psi}, \tau_{yx} = -2870 \text{ psi}$$

Then a line was drawn between them, crossing the σ-axis at *O*. The value of the stress at *O* is





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Example 4-2

$$\sigma_{avg} = (\sigma_x + \sigma_y)/2 = (8030 + 0)/2 = 4015 \text{ psi}$$

Step 7. We compute the values for *a*, *b*, and *R* from

$$a = (\sigma_x - \sigma_y)/2 = (8030 - 0)/2 = 4015 \text{ psi}$$
$$b = \tau_{xy} = 2870 \text{ psi}$$
$$R = \sqrt{a^2 + b^2} = \sqrt{(4015)^2 + (2870)^2} = 4935 \text{ psi}$$

Step 8. Figure 4–17 shows the completed Mohr's circle. The circle has its center at *O* and the radius *R*. Note that the circle passes through the two points originally plotted. It must do so because the circle represents all possible states of stress on the element *K*.

Step 9. The maximum principal stress is at the right side of the circle.

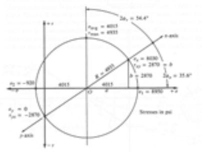
$$\sigma_1 = \sigma_{avg} + R$$
$$\sigma_1 = 4015 + 4935 = 8950 \text{ psi}$$

Step 10. The minimum principal stress is at the left side of the circle.

$$\sigma_2 = \sigma_{avg} - R$$
$$\sigma_2 = 4015 - 4935 = -920 \text{ psi}$$

Step 11. At the top of the circle,

$$\sigma = \sigma_{avg} = 4015 \text{ psi}$$
$$\tau = \tau_{max} = R = 4935 \text{ psi}$$



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Example 4-2

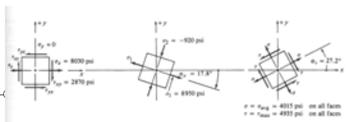

FIGURE 4–18 Results from Mohr's circle analysis

The value of the normal stress on the element that carries the maximum shear stress is the same as the coordinate of *O*, the center of the circle.

Step 12. Compute the angle $2\phi_\sigma$ and then ϕ_σ. Use the circle as a guide.

$$2\phi_\sigma = \arctan(b/a) = \arctan(2870/4015) = 35.6^\circ$$
$$\phi_\sigma = 35.6/2 = 17.8^\circ$$

Note that ϕ_σ must be measured *clockwise* from the original *x*-axis to the direction of the line of action of σ_1 for this set of data. The principal stress element will be rotated in the same direction as part of step 14.

Step 13. Compute the angle $2\phi_s$ and then ϕ_s. From the circle we see that

$$2\phi_s = 90^\circ - 2\phi_\sigma = 90^\circ - 35.6^\circ = 54.4^\circ$$
$$\phi_s = 54.4/2 = 27.2^\circ$$

Note that the stress element on which the maximum shear stress acts must be rotated *counterclockwise* from the orientation of the original element for this set of data.

Step 14. Figure 4–18 shows the required stress elements. They are identical to those shown in Figure 4–11.

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