Chapter 5
Uncertainty Analysis

Uncertainty Analysis
- Error = difference between true value and observed
- Uncertainty = we are estimating the probable error, giving us an interval about the measured value in which we believe the true value must fall.
- Uncertainty Analysis = process of identifying and qualifying errors.

Measurement Errors
- Two General Groups
  1. Bias Error - shifts away from true mean
  2. Precision Error - creates scatter
**Figure 5.1**

- **Best estimate of true value:** $X' = X \pm \mu_x$
- **Assumption:**
  1. Test objective known
  2. Measurement is clearly defined, where all bias errors have been compensated through calibration
  3. Data obtained under fixed operating conditions
  4. Engineers have experience with system components

---

**Design-Stage Uncertainty Analysis**

Design-stage uncertainty analysis refers to an initial analysis performed prior to the measurement. It is useful for selecting instruments, selecting measurement techniques, and obtaining an approximate estimate of the uncertainty likely to exist in the measured data. At this point, the measurement system and associated procedures are but a concept. Usually little is known about the instruments, and in many cases they are still pictures in a catalog. Major facilities may need to be built and equipment ordered with considerable lead time. Uncertainty analysis should be used to assist in the selection of equipment and procedures based on their relative performance and even cost. In the design stage, distinguishing between bias and precision errors is too difficult to be of concern. Instead, consider only sources of uncertainty in general.

---

**Two Basic Types of Error Considered**

- **Zero-Order Uncertainty** – $\mu_0$ is an estimate of the expected uncertainty caused by reading the data (interpretation error or quantization error). It is assumed that this error is less than the instrumentation error.
  - **Arbitrary Rule** – set $\mu_0$ equal to ½ instrument resolution with 95% probability.
  - $\mu_0 = \pm \frac{1}{2}$ resolution (95%)
  - At 95%, only 1 in 20 measurements would fall outside the interval defined by $\mu_0$. 
**Two Basic Types of Error Considered**

- **Instrument Error** - "µc", is a combination of all component errors and gives an estimate of the instrument bias error. Errors are combined by the root-sum-squares (RSS) method.
  - Elemental errors combine to give an increase in uncertainty
    \[
    \mu_x = \pm \sqrt{e_1^2 + e_2^2 + \ldots + e_k^2}
    \]
  - Must maintain consistency in units of error
  - Should use the same probability level in all cases (95% preferred)

- Assume that the combined errors follow a Gaussian distribution. By using RSS, we can get a statistical estimate of error, which assumes that the worst-case element errors will not all add up to the highest error at any one time.

**Design Stage Uncertainty**

- Combine zero order and instrumental error
  - Estimates minimum uncertainty
  - Assumes perfect control over test conditions and measurement procedures
  - Can be used to chain sensors and instruments to predict overall measurement system error

---

**Figure 5.2** Design stage uncertainty procedure in combining uncertainties. Figoliola, 2000
Error Sources

- Calibration Errors
- Data-acquisition errors
- Data-reduction errors

Within each category, there are elemental error sources. It is not critical to have each elemental error listed in the right place. It is simply a way to organize your thinking. The final uncertainty will come out the same.

Calibration Error

<table>
<thead>
<tr>
<th>Table 4.1 Calibration Error Source Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error Source 1</td>
</tr>
<tr>
<td>Primary to instr</td>
</tr>
<tr>
<td>Calibr to instr lab standard</td>
</tr>
<tr>
<td>Lab standard to instr standard</td>
</tr>
</tbody>
</table>

Sources:
1. Bias and precision error in standard used
2. Manner in which standard is applied
3. Calibration does not eliminate error; it reduces it to a more acceptable level.
Bias and Precision Errors

- In general, the classification of an error element as containing bias error, precision error, or both can be simplified by considering the methods used to quantify the error.
- Treat an error as a precision error if it can be statistically estimated in some manner; otherwise treat the error as a bias error.

Bias Error

- A bias error remains constant during a given series of measurements under fixed operating conditions.
- Thus, in a series of repeated measurements, each measurement would contain the same amount of bias.
- Being a fixed value, the bias error cannot be directly discerned by statistical means alone.
Bias Error
- It can be difficult to estimate the value of bias or in many cases, recognize the presence of bias error.
- A bias error may cause either high or low estimates of the true value.
- Accordingly, an estimate of the bias error must be represented by a bias limit, noted as ±B, a range within which the true value is expected to lie.

Bias Error
- Bias can only be estimate by comparison.
- Various methods can be used:
  - Calibration
  - Concomitant methodology
  - Interlaboratory comparisons
  - Experience

Bias Error
- The most direct method is by calibration using a suitable standard or using calibration methods of inherently negligible or small bias.
- Another procedure is the use of concomitant methodology, which is using different methods of estimating the same thing and comparing the results.
Bias Error

- An elaborate but good approach is through interlaboratory comparisons of similar measurements, an excellent replication method.
- In lieu of these methods, a value based on experience may have to be assigned.

Precision Error

- When repeated measurements are made under nominally fixed operating conditions, precision errors will manifest themselves as scatter of the measured data.

Precision Error

- Precision error is affected by:
  - Measurement System (repeatability and resolution)
  - Measured Variable (temporal and spatial variations)
  - Process (variations in operating and environmental conditions)
  - Measurement Procedure and Technique (repeatability)
Error Propagation

- Consider: \( y = f(x) \)
- True \( x \): \( \bar{x} = \pm tSx \)
- True \( y \): \( \bar{y} \pm \delta y = f(\bar{x} \pm tSx) \)

\[ x \text{ is measured to establish its sample mean and precision.} \]

Expanding as Taylor Series

\[ \bar{y} \pm \delta y = f(\bar{x}) \pm \left[ (dy/dx)_{\bar{x}} \pm tS\bar{x} \right] \quad \text{By inspection} \]

\[ \bar{y} = f(\bar{x}) \quad \text{and} \quad \pm \delta y = \pm \left[ (dy/dx)_{\bar{x}} \pm tS\bar{x} \right] \]

A linear approximation that ignores high order terms:

\[ \delta y = (dy/dx)_{\bar{x}} \pm tS\bar{x} \]

The derivative \( (dy/dx)_{\bar{x}} \pm tS\bar{x} \) defines sensitivity at \( \bar{x} \) the width of interval \( tS\bar{x} \) defines precision interval about \( y \) that is \( \pm \delta y \).

From this, the uncertainty in \( y \) is dependent on uncertainty in \( x \):

\[ u_y = (dy/dx)_{\bar{x}} = \pm u_x \]

Expanding as Taylor Series

- In multivariable applications,
- \( R = f(x_1, x_2, \ldots, x_L) \)
- \( L = \text{number of independent variables} \)
  
  Best estimate of true value \( R' \):
  
  \[ R' = \bar{R} \pm u_R \ (P\%) \]

  Sample mean:
  
  \[ \bar{R} = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_L) \]

  Uncertainty in \( R \):
  
  \[ u_R = f(\pm u_{x_1}, u_{x_2}, \ldots, u_{x_L}) \]
Uncertainty in R

- $U_R$ represents the uncertainty of each independent variable as it is propagated through to the result $R$. The estimate of $U_R$ came from Kline-McClintock.
- Using a Taylor Series expansion of "R", a sensitivity index: $\theta = \frac{\partial R}{\partial x_i} \bigg|_{i=1, 2, \ldots, L}$
  Index is evaluated using either mean or expected nominal values of $x$.

Uncertainty Analysis: Error Propagation

- The sensitivity index relates to how changes in each $x_i$ affect $R$.
- The use of the partial derivative is necessary when $R$ is a function of more than one variable.
- This index is evaluated by using either the mean values or, lacking these estimates, the expected nominal values of the variables.

Uncertainty Analysis: Error Propagation

- From a Taylor series expansion, the contribution of the uncertainty in $x$ to the result, $R$, is estimated by the term $\theta_j u_{x_j}$
- The propagation of uncertainty in the variables to the result will yield an uncertainty estimate given on the next slide.
The Uncertainty Estimate

\[ u_R = \pm \sqrt{\sum_{i=1}^{L} (\theta_i u_{x_i})^2} \]

Sequential Perturbation

- This is the numerical approach to estimate the propagation of uncertainty, using finite difference.
  - replaces the direct approach that requires the solution of cumbersome partial differential equations.
- The method is straightforward and uses a finite-difference method to approximate the derivatives.

Method

1. Based on measurement of independent variables, calculate
   - \( R_0 = f(x_1, x_2, \ldots, x_L) \)
2. Increase \( x_i \) by \( u_{x_i} \) "uncertainty" to get \( R_i^+ \)
   - \( R_{i+} = f(x_1 + u_{x_1}, x_2, \ldots, x_i) \)
   - \( R_{i+} = f(x_1, x_2 + u_{x_2}, \ldots, x_i) \)
   - \( R_{i+} = f(x_1, x_2, \ldots, x_i + u_{x_i}) \)
3. Repeat by decreasing \( x_i \) by \( u_{x_i} \) to get \( R_i^- \)
4. Calculate difference \( \delta_{R_i^+} \) and \( \delta_{R_i^-} \) for \( i = 1, 2, \ldots, L \)
   - \( \delta_{R_i^+} = R_i^+ - R_0 \)
   - \( \delta_{R_i^-} = R_i^- - R_0 \)
Method

5. Evaluate the approximation of uncertainty contribution from each variable.
\[ \delta_{R_i} = \left( \delta_{R_i}^2 + \delta_{R_i}^{-2} \right)/2 \approx \delta_{\sum_i} \]

6. Uncertainty
\[ u_R = \pm \left[ \sum_{i=1}^{L} (\delta_{R_i})^2 \right]^{1/2} \quad (\%\text{P}) \]

Numerical approach and partial diff eq approach give similar results. Occasional unreasonable estimates of error propagation can be caused by rapidly changing sensitivity indexes due to small change in independent variables.

Advanced Stage and Single Measurement Uncertainty

- Considers not only resolution errors and intrinsic instrument errors, but also procedural and test control errors
- Method:
  - Estimates expected uncertainty beyond design stage
  - Reports results of a test program over range of one or more parameters with no or few replications at each test condition.

Advanced Stage and Single Measurement Uncertainty

- In design-stage uncertainty analysis, only errors caused by a measurement system’s resolution and estimated intrinsic errors are considered.
- Single-measurement uncertainty analysis permits taking that treatment further by considering procedural and test control errors that affect the measurement.
Advanced Stage and Single Measurement Uncertainty

- Goal:
  - Estimate the uncertainty in some measured value of x or result R, based on estimation of uncertainty in the factors that influence x or R.

- Zero-Order:
  - All variables and parameters are fixed, except for the physical act of observation or quantization.
  - Instrument resolution is the only factor.
  - $u_0 = 1/2$ resolution

Higher Order Uncertainty

- Consider controllability of test operating conditions
  - First order $u_1$ evaluates influence of time
    - What is variability if all factors are held constant and you collect data over period of time?
    - Collect $N \geq 30$ samples, $u_1 \pm t_{0.95} \sigma_{u_1}$
    - If $u_1 = u_0$, then time has no influence

- Higher Order
  - Each successive order identifies a factor that influences the outcome, thus giving a more realistic estimate of uncertainty
  - At each stage, collect $N \geq 30$ samples,
    $u_i = t_{0.95} \sigma_{u_i}$
  - For example, at the second level it might be appropriate to assess the limits of the ability to duplicate the exact operating conditions and the consequences on the test outcome.
Higher Order Uncertainty

- Nth Order
  - Instrument calibration uncertainty $u_L$ is entered in
    $$u_L = \sqrt{(u_1)^2 + \sum_{i=1}^{N-1} u_i^2}$$ (95%)
  - Allows for direct comparison between results obtained by either different instruments or different test facilities.
  - Approximate value to use in reporting results.

- Note the difference between design stage and advanced stage is $u_1, u_2, \ldots, u_{N-1}$ terms that consider the influence of time and operating conditions.

Multiple Measurement Uncertainty

- Estimation of uncertainty assigned to measured variable, based on set of measurements obtained under fixed operating conditions.

- Parallels uncertainty standards approved by many professional societies.
Method
1. Identify elemental error in three groups (calibration, data acquisition, and data reduction)
2. Estimate the magnitude of bias and precision error for each elemental error.
3. Estimate propagation through to results

Method
- consider \( x \) to have precision error \( P_{ij} \) and bias \( B_{ij} \)
  - \( i = 1 \) for calibration
  - \( i = 2 \) for data acquisition
  - \( i = 3 \) for data reduction

Then \( e_{ij} = P_{ij} + B_{ij} \) is the elemental error for the \( i \)th group and \( j \)th elemental error.

\[
P_{ij} = t_{x,0.95}s_x
\]

1. Propagation of precision errors in the source group is called source precision index \( P_i \), estimated by RSS.
2. Measurement precision index is a basic measurement of the precision of the overall measured variable due to all elemental errors.
3. Source bias treated similarly.
4. Measurement bias represents the basic measurement of the elemental errors that affect the overall bias.
5. Measurement uncertainty in \( x \) can be represented as a combination of bias and precision uncertainty.
Estimation of the degrees of freedom, v, in the precision index, P, requires special treatment, since elements have different degrees of freedom.

In this case, the degrees of freedom in the measurement precision index is estimated by using Welch-Satterthwaite Formula:

\[
\nu = \frac{\left(\sum_{i=1}^{3} \sum_{j=1}^{K} P_{ij}^2 \right)^2}{\sum_{i=1}^{3} \sum_{j=1}^{K} \frac{P_{ij}^4}{\nu_{ij}}}
\]

Where i refers to the three source error groups and j to each Elemental error within each source group with \(\nu_{ij} = N_{ij} - 1\).