Chapter 3
Measurement System Behavior
Part 2

1st Order – Sine Function Input

• Examples of Periodic: vibrating structure, vehicle suspension, reciprocating pumps, environmental conditions
• The frequency of the input significantly affects measuring system time response.
• Consider:

\[ \tau \dot{y} + y = KA \sin \omega t \quad t \geq 0^* \]
\[ y(0) = y_0 \]

General Solution

\[ y(t) = Ce^{-\omega t} + \frac{KA}{1 + (\omega \tau)^2} \sin(\omega t - \tan^{-1} \frac{t}{\tau}) \]

Solve for C at t=0, y(0)=y_0

Output Transient

\[ Ce^{-\omega t} \to 0 \quad \text{With t going past } 5\tau \]
\[ \omega = 2\pi f \]

Note: amplitude and phase shifts are dependent on the frequency of input.
**Output Steady State**

- Amplitude and phase shift are functions of input frequency
- Output frequency is the same as input frequency
- Output remains as long as the forcing function

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**Rewritten:**

\[ y(t) = Ce^{-t/\tau} + B(w)\sin[w(t) + j(w)] \]

**Magnitude:**

\[ B(\omega) = (KA)/[1 + (\omega\tau)^2]^{1/2} \]

**Phase Shift:**

\[ \varphi(\omega) = -\tan^{-1}\omega\tau \]

Where \( \tau \) is the time constant.

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**Time Delay**

\[ \beta_1 = (\varphi(\omega))/\omega \]

Value of \( \beta_1 \) is negative, indicating a time delay between input and output.

Magnitude ratio of input/output:

\[ m(\omega) = B/KA = 1/(1+(\omega\tau)^2)^{1/2} \]

Note the frequency dependency.
For value of $\omega \tau$ which gives $m(\omega)$ near unity. The first order MS will transmit nearly all of the input signal with little delay or attenuation of signal.

$B(\omega) = KA, \phi(\omega) = 0^\circ$

If you want to monitor an input signal with high frequency, you will need $\tau$ to be very small in order to give $m(\omega) = 1$

Dynamic Error

$\delta(\omega) = m(\omega) - 1$

– measures systems inability to adequately reconstruct the amplitude of input at a given frequency.

• Want to minimize $\delta(\omega)$

• Perfect reproduction is not possible

Frequency Bandwidth

• Frequency band over which $m(\omega) \geq 0.707$ or where $dB = 20 \log m(\omega)$ does not drop more than –3 dB between $\omega_n$ and $\omega_m$ for a given $\tau$

  $dB = 20 \log m(\omega_n) + 20 \log m(\omega_m)$
  $dB = 20 \log (1) + 20 \log (0.4)$
  $dB = -7.95$
**Determination of Frequency Response**

- The frequency response is found by dynamic calibration, by applying a simple periodic waveform input to the sensor stage and monitoring the output. However, this can be impractical in some physical systems.
  - Model system frequency response
  - Calibrate components independently and then combine them to develop system.
    - Ex: (load cell and transmitter) calibration
    - Range 0 to 40,000 lbs

**Determination of Frequency Response con’t:**

- Load cell is calibrated in factory
- 4-20MA transmitter calibrated using an analog signal represents the range of the load cell – 0 to 30mv/volt
   - Load cell and transmitter are loaded with test weights to confirm approximate range (0 to 30,000 lbs max available)

**2nd Order System**

- Possesses inertia and contains a second deviative term, such as accelerometers, diaphragm pressure transducers, and acoustic microphones.
  \[ a_2 \dddot{y} + a_1 \ddot{y} + a_0 y = F(t) \quad \text{or} \]
  \[ \left( \frac{1}{\omega_n^2} \right) \ddot{y} + (2\zeta \frac{\ddot{y}}{\omega_n}) + y = KF(t) \]
  where \( \omega_n = \sqrt{a_0 / a_2} \) = natural frequency
2nd Order System

- \( \zeta = \frac{\alpha}{2(\alpha_0 a_t)^2} \) Zeta = damping ratio

- The damping ratio is a measure of system damping, a property of a system that enables it to dissipate energy internally.

Homogeneous solution

- Quadratic equations have two roots
  \[ \frac{1}{(w_n^2)}\lambda^2 + (2\zeta/w_n)\lambda + 1 = 0 \]
  \( \lambda_1, \lambda_2 \)

- \( \lambda_1, \lambda_2 = -\zeta w_n \pm w_n\sqrt{\zeta^2 - 1} \)

- Homogeneous solution gives us the transient response
- Finds the roots of the characteristic equation

- The three forms of homogeneous solution depend on the value of damping
- \( 0 < \zeta < 1 \) (underdamped) – oscillatory response
  \( y_a(t) = Ce^{-\zeta\omega_n t} \sin(w_n\sqrt{1 - \zeta^2}t + \theta) \)
- \( \zeta = 1 \) (critically damped – asymptotically approaches SS)
  \( y_a(t) = Ce^{\lambda_1 t} + C_2 e^{\lambda_2 t} \)
- \( \zeta > 1 \) (overdamped)
  \( y_a(t) = Ce^{\lambda_1 t} + C_2 e^{\lambda_2 t} \)
Step Function Response

- Response equation given in text – 3.1Ss-3.1Sc
  \( \zeta=1 \) (critically damped)
  \( Y(t)=KA-KA(1+\omega_n t)e^{-\omega_nt} \)
- For an underdamped system, the transient response is oscillatory, with a periodic behavior of period
  \( T_d = \frac{2\pi}{\omega_n} \),
  with ringing frequency
  \( \omega_d = \omega_n\sqrt{1-\zeta^2} \)

Note: this is dependent on the instrument, not the signal.

Ringing

- In instruments, this oscillatory behavior is called “ringing.”
- The ringing phenomenon and the associated ringing frequency are properties of the measurement system and are independent of the input signal.
- It is the free oscillation frequency of a system displaced from its equilibrium.

1. Duration of transient response controlled by \( \zeta \omega_n \). For \( \zeta>1 \), the response \( y_\infty \rightarrow KA \) at \( t \rightarrow \infty \), but for larger \( \zeta \omega_n \) the response is faster.
2. Time required to reach 90% of step input \( Au(t)=KA-y_0 \) is called rise time. Rise time is decreased by decreasing the damping ratio \( \zeta \).
3. Time to reach ±10% of steady state is called settling time for oscillatory systems.

Note: a faster rise may not necessarily reach a steady state faster if the oscillations are large.
**Sine Function Input**

- Response of 2nd order system to $F(t) = A \sin \omega t$
  
  $y(t) = y_h + \left[ KA \sin(\omega t + \phi(\omega)) \right] \frac{\left[ 1 - (\omega / \omega_n)^2 \right]^2 + (2 \zeta \omega / \omega_n)^2}{12}$

- Frequency dependent phase shift
  
  $\phi(\omega) = -\tan^{-1}(2\zeta \omega / \omega_n) / (1 - (\omega / \omega_n)^2)$

Exact form of $y_h$ depends on ($\zeta$) damping ratio

*Note: $h =$ homogeneous solution*

**Steady State Response**

- $Y_{steady}(t) = \beta(\omega) \sin(\omega t + \phi(\omega))$

- Amplitude:
  
  $B(\omega) = KA / \left[ 1 - (\omega / \omega_n)^2 \right]^2 + (2 \zeta \omega / \omega_n)^2 \right]^{1/2}$

- The amplitude of the output signal from a second-order measurement system is frequency dependent.
Magnitude Ratio

- Magnitude Ratio:
  \[ m(\omega) = \frac{B}{KA} \]
- \( \omega_n \) is a function of the measurement system
- \( \omega \) is a function of the input signal

Magnitude Ratio

- Fig 3.16 and Fig 3.17 in text demonstrate magnitude and phase as functions of \( \omega/\omega_n \)
- In ideal system, \( m(\omega) = 1.0 \) and \( \phi(\omega) = 0 \)
- In general, as \( \omega/\omega_n \) gets large, \( m(\omega) \rightarrow 0 \) and \( \phi(\omega) \rightarrow -\pi \)

Phase Shift
Resonance Frequency

\[ \omega_r = \omega_n \sqrt{1 - 2\zeta^2} \]

- It is a property of MS, operating near resonance frequency. It can damage or distort either the data or the instrument.
- When \( \omega = \omega_n \), \( m(\omega) \to \infty \), and \( \phi(\omega) \to -\pi \), it occurs for the underdamped system \( \zeta = 0 \). It is called the resonance band.
- Systems with damping \( \zeta > 0.7 \) do not resonate.

Resonance Frequency cont:

- At low \( \omega/\omega_n \), \( m(\omega) \approx 1 \) and \( \phi(\omega) \approx 0 \). This is called the transmission band, which is defined by \( 3\text{dB} \geq m(\omega) \geq -3\text{dB} \).
  - Here we have a representation of the dynamic signal content.
- At large \( \omega/\omega_n \), \( m(\omega) \to 0 \), which is called the filter band
  - \( m(\omega) \leq -3\text{dB} \)
  - Here we lose high frequency signal content, which is good only if you want low frequency information!
Multiple-Function Inputs

- When models are used that are linear, ordinary differential equations subjected to inputs that are linear in terms of the dependent variable, the principle of superposition of linear systems will apply to the solution of these equations.

Principle of Superposition

- The theory of superposition states that a linear combination of input signals applied to a linear measurement system produces an output signal that is simply the linear addition of the separate output signals that would result if each input term had been applied separately.

Principle of Superposition

- The forcing function of a form:
  \[ F(t) = A_0 + \sum_{i=1}^{N} A_i \sin(\omega_i t) \]
  is applied to a system, then the combined steady response will have the form:
  \[ KA_0 + \sum_{n=1}^{N} B(\omega_n) \sin(\omega_n t + \phi(\omega_n)) \]
  Where \( B(\omega_n) = KA_n M(\omega_n) \)
Coupled Systems

- When a measurement system consists of more than one instrument, the measurement system behavior can become more complicated.
- As instruments in each stage of the system are connected, the output from one stage becomes the input to the next stage and so forth.

Such measurement systems will have an output response to the original input signal that is some combination of the individual instrument responses to the input.
- The system concepts of zero-, first-, and second-order systems studied previously can be used for a case-by-case study of the coupled measurement system.
- This is done by considering the input to each stage of the measurement system as the output of the previous stage.

![Diagram of coupled systems](image)

*Figure 3.24 Coupled systems, describing the system transfer function.*
Coupled Systems

• The previous slide depicts a measurement system consisting of H interconnected devices, \( j = 1, 2, \ldots, H \), each device described by a linear system model.

Coupled Systems

• The overall transfer function of the combined system, \( G(s) \), will be the product of the transfer functions of each of the individual devices, \( G_j(s) \), such that:
  \[ KG(s) = K_1G_1(s)K_2G_2(s)\ldots K_HG_H(s) \]

• The overall system static sensitivity is described by:
  \[ K = K_1K_2K_3\ldots K_H \]

• The overall system magnitude ratio will be the product:
  \[ M(\omega) = M_1(\omega)M_2(\omega)\ldots M_H(\omega) \]

• The overall system phase shift will be the sum:
  \[ \phi(\omega) = \phi_1(\omega) + \phi_2(\omega) + \ldots + \phi_H(\omega) \]